Exercise 2.5.3

Consider the equation $\dot{x} = rx + x^3$, where $r > 0$ is fixed. Show that $x(t) \to \pm\infty$ in finite time, starting from any initial condition $x_0 \neq 0$.

Solution

Show that $x(t)$ blows up in finite time by solving the initial value problem.

$$\frac{dx}{dt} = rx + x^3, \quad x(0) = x_0 \neq 0, \quad r > 0$$

Separate variables.

$$\frac{dx}{rx + x^3} = dt$$

Integrate both sides.

$$\int^x \frac{dz}{rz + z^3} = t + C$$

Use partial fraction decomposition on the left side.

$$\int^x \frac{dz}{z(r + z^2)} = t + C$$

$$\int^x \left[\frac{1}{rz} - \frac{z}{r(r + z^2)}\right] dz = t + C$$

$$\frac{1}{r}\left(\int^x \frac{dz}{z} - \int^x \frac{z}{r + z^2}\, dz\right) = t + C$$

Make the following substitution in the second integral.

$$u = r + z^2$$

$$du = 2z\, dz \quad \to \quad \frac{du}{2} = z\, dz$$

Consequently,

$$\frac{1}{r}\left[\ln|x| - \int^{r+x^2} \frac{1}{u}\left(\frac{du}{2}\right)\right] = t + C$$

$$\frac{1}{r}\left(\ln|x| - \frac{1}{2}\int^{r+x^2} \frac{du}{u}\right) = t + C$$

$$\frac{1}{r}\left(\ln|x| - \frac{1}{2}\ln|r + x^2|\right) = t + C$$

$$\frac{1}{r}\left(\ln|x| - \ln\sqrt{r + x^2}\right) = t + C$$

$$\frac{1}{r}\ln\frac{|x|}{\sqrt{r + x^2}} = t + C.$$

Apply the initial condition $x(0) = x_0$ to determine C.

$$\frac{1}{r}\ln\frac{|x_0|}{\sqrt{r+x_0^2}} = 0 + C \quad \rightarrow \quad C = \frac{1}{r}\ln\frac{|x_0|}{\sqrt{r+x_0^2}}$$

So the solution becomes

$$\frac{1}{r}\ln\frac{|x|}{\sqrt{r+x^2}} = t + \frac{1}{r}\ln\frac{|x_0|}{\sqrt{r+x_0^2}}$$

$$\frac{1}{r}\left(\ln\frac{|x|}{\sqrt{r+x^2}} - \ln\frac{|x_0|}{\sqrt{r+x_0^2}}\right) = t$$

$$\frac{1}{r}\ln\frac{|x|}{\sqrt{r+x^2}}\frac{\sqrt{r+x_0^2}}{|x_0|} = t$$

$$\ln\frac{|x|}{\sqrt{r+x^2}}\frac{\sqrt{r+x_0^2}}{|x_0|} = rt$$

$$\frac{|x|}{\sqrt{r+x^2}}\frac{\sqrt{r+x_0^2}}{|x_0|} = e^{rt}$$

$$\frac{x^2}{r+x^2}\frac{r+x_0^2}{x_0^2} = e^{2rt}$$

$$\frac{1}{\frac{r}{x^2}+1} = \frac{x_0^2e^{2rt}}{r+x_0^2}$$

$$\frac{r}{x^2}+1 = \frac{r+x_0^2}{x_0^2e^{2rt}}$$

$$\frac{r}{x^2} = \frac{r+x_0^2}{x_0^2e^{2rt}} - 1$$

$$\frac{r}{x^2} = \frac{r+x_0^2-x_0^2e^{2rt}}{x_0^2e^{2rt}}$$

$$\frac{x^2}{r} = \frac{x_0^2e^{2rt}}{r+x_0^2-x_0^2e^{2rt}}$$

$$x^2 = \frac{rx_0^2e^{2rt}}{r+x_0^2(1-e^{2rt})}.$$

Therefore, taking the square root of both sides,

$$x(t) = \pm\sqrt{\frac{rx_0^2e^{2rt}}{r+x_0^2(1-e^{2rt})}}.$$

Note that if $x_0 = 0$, then $x(t) = 0$; it would take an infinite amount of time for $x(t)$ to blow up.

If $x_0 \neq 0$, then $x(t)$ blows up when the denominator is zero.

$$r + x_0^2(1 - e^{2rt}) = 0$$

$$x_0^2(1 - e^{2rt}) = -r$$

$$1 - e^{2rt} = -\frac{r}{x_0^2}$$

$$-e^{2rt} = -\frac{r}{x_0^2} - 1$$

$$e^{2rt} = \frac{r}{x_0^2} + 1$$

$$\ln e^{2rt} = \ln\left(\frac{r}{x_0^2} + 1\right)$$

$$2rt = \ln\left(\frac{r}{x_0^2} + 1\right)$$

$$t = \frac{1}{2r}\ln\left(\frac{r}{x_0^2} + 1\right)$$

Therefore, $x(t) \to \pm\infty$ in finite time, starting from any initial condition $x_0 \neq 0$.